Int. J. Solids Structures Vol. 29, No. 14/15, pp. 1837–1847, 1992
Printed in Great Britain.

0020-7683/92 $5.00 + .00
Pergamon Press plc

ELASTIC MODULI OF PERFORATED PLATES IN THE NEIGHBORHOOD OF CRITICAL STATE

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Abstract—Percolation theory was applied to determine the critical surface area density for a random distribution of circular voids in a two-dimensional elastic medium. Additionally, the percolation theory provides the scaling laws characterizing nonlinear dependence of elastic moduli on lacunity in the neighborhood of critical state. It is also argued that the percolation theory complements the traditional effective continua models providing a measure of their accuracy.

INTRODUCTION

An important class of inelastic deformation processes, typical of metals, metallic alloys and some ceramics, is characterized by extensive micro-cavitation over a large fraction of the specimen volume. During the latter phase of their growth cavities may coalesce, forming clusters large enough to compromise the integrity of the specimen on the macro-scale. The rapid and highly localized (non-uniform) process of void coalescence leading to ductile rupture has been studied in the past by many authors (see, for example, Čadek, 1987; Riedel, 1987; Tvergaard, 1987; Becker *et al.*, 1988).

The fracture mode of cavitating materials depends on many factors, among which the initial porosity, temperature and stress levels are commonly considered to be dominant. At elevated temperatures and for protracted load durations strain to rupture dramatically decreases. This loss of ductility is directly attributable to the increasing porosity caused by the creep cavitation. The coalescing cavities eventually form a large void cluster leading to fractures characterized by dimpled, intergranular surfaces.

The exact micro-mechanisms causing cavitation have been reviewed in detail by Riedel (1987) and others and are considered beyond the scope of this study. The objective of this paper is restricted to the determination of the elastic moduli of an otherwise homogeneous material containing a random distribution of voids. Even more specifically, the attention is focused on the determination of critical porosity signalling the onset of macro-failure.

The problem of determination of compliance of a solid weakened by an ensemble of micro-defects attracted considerable attention in the last decade. Most of the models were developed within the framework of the first-order effective continua (mean field) theories which are, in general, applicable to low-to-moderate micro-defect densities.

In order to predict the onset of failure it is important to develop analytical estimates of the critical lacunity at which the macro-stiffness of the material vanishes. This task may be addressed within the framework of the percolation theory. The objective of the present paper is to explore the applicability of the percolation theory to the problem of deter-

mination of the compliance of an elastic plate containing a random distribution of circular voids for the entire range of densities.

EFFECTIVE CONTINUA MODELS

Analytical models relating the macro-response of a specimen to the micro-structure of a material are commonly labeled as micro-mechanical. The micro-to-macro mapping involves introduction of the representative volume element (RVE) defined as a statistically representative part of the volume (Hill, 1967; Nemat-Nasser and Hori, 1990) mapping on a material point of the effective continuum. Thus, by definition, the number N of micro-defects within the RVE must be large. The exact determination of the stress and strain fields within the RVE weakened by N interacting micro-defects is a nontrivial problem since it involves solution of N coupled integral equations (Kunin, 1983). The kernels of these integrals are defined via Green's displacement tensor which is available only for simple materials and simple defect geometries.

Thus, for a spatially homogeneous and random field of defects the "exact" solution is not a feasible alternative. A set of the so-called effective continua models was, therefore, suggested in order to provide approximate solutions of the problem. Most of these models belong to the class of the first-order theories based on the premise that the mean values of the stress, strain and damage fields suffice for the determination of the response. In effect, these models assume that: (a) the external fields of each defect weakly depend on the exact position of adjacent defects and inhomogeneities, and (b) the external stress field of each defect (inhomogeneity) is equal to the far-field (macro) stress. As a result, each defect can be analyzed as an isolated inclusion placed in the center of the RVE containing homogenized, effective material. The parameters of the effective material are typically determined from energy considerations.

The simplest (Taylor) model, valid only for very dilute micro-defect concentrations, is based on the assumption that each defect is surrounded by the original (virgin) material completely ignoring the existence of other micro-defects. The classical self-consistent model was for the problem we are considering used first by Vavakin and Salganik (1975) and Budiansky and O'Connell (1976), and subsequently applied by Horii and Nemat-Nasser (1983, 1986), Sumarac and Krajcinovic (1987) and Krajcinovic and Sumarac (1989). An interesting improvement (known as double-embedding or three-phase model) of the self-consistent model was suggested by Christensen and Lo (1979). The so-called differential method was applied to this class of problems by Vavakin and Salganik (1975), Cleary *et al.* (1980) and Hashin (1988).

In the case of homogeneous macro-stresses and macro-strains, isotropic and linear elastic matrix and defects of simple and regular geometries, the above-mentioned models require acceptable levels of numerical effort needed to determine elastic parameters as a function of the microcrack density. Taylor's model, in fact, often leads to closed-form, analytical solutions. However, as expected, different methods lead to different estimates for elastic moduli. The differences are especially significant for larger micro-defect densities (see Sumarac and Krajcinovic, 1987 and Nemat-Nasser and Hori, 1990). Moreover, it was shown that these methods lead to different limits at inclusion saturations (Christensen, 1990). Thus, despite significant progress in modeling, a considerable doubt persists with regard to the applicability of these methods for larger micro-defect concentrations. Moreover, faced with a host of different models it becomes difficult to select the most appropriate method for the considered case.

Eliminating from consideration the distance between the defects, the first-order effective continua (mean-field) models are rendered local and, therefore, unsuitable for analyses of deformation processes dominated by direct defect interaction (see, for example, Green, 1940, or Kouris and Tsuchida, in press). In the case when the distances separating the micro-defects are small they have dominant influence on the stress field (fluctuations from the mean value) and must be therefore included in computations. The response is, therefore, in the case of many defects dominated by the highest statistical momenta of the micro-defect distribution (Duxbury, 1990). Considerations of the direct micro-crack interaction

(Kachanov, 1987; Kachanov and Laures, 1989; Ju and Chen, unpublished) lead to computationally intensive models despite an array of simplifying assumptions introduced to increase the tractability. In most cases application of these models requires special software. Nevertheless, they provide valuable data regarding the prevailing trends at growing micro-crack densities. Rigorous higher-order theories (Kanaun, 1977) become totally unmanageable in applications.

The state-of-art in development and application of mean-field models was aptly summarized in Cleary *et al.* (1980) who stated that "the problem of multiple adjacent inclusions appears to be prohibitively complex" especially since it is not clear "how much information we need and whether or not all effects are really important". Even the second-order theories, such as one suggested by Chatterjee and Mal (1978), "take no account of possible clustering effects that might develop". With considerable foresight Cleary *et al.* (1980) conclude that "until some tractable probabilistic description has been achieved, it seems that a single isolated site model of heterogeneities will have to suffice".

For the already stated reasons one of the most interesting segments of the macro-response is one just preceding the transition from a state characterized by a diffuse field of small isolated defects to a state dominated by a single defect cluster spanning not one but many RVEs and possibly the entire specimen. In the proximity of the critical regime, defined as a state of vanishing stiffness, the local stress fluctuations vastly exceed the macro-stress levels.

PERCOLATION THEORY

Applications of conventional continuum models for the determination of macro-parameters characterizing transport processes through strongly disordered solids containing a large density of micro-inhomogeneities (inclusions) are typically limited to:

(a) low-to-moderate concentrations of inclusions (defects, pores, second-phase particles, micro-cracks, etc.),
(b) elastic and isotropic matrices,
(c) simple inclusion geometries (shapes), and/or in some cases
(d) regular, en echelon, spatial distributions of inclusions.

Hence, to retain the computational efficiency, the micro-structural disorder is in some general sense approximated by an "effective" order. While this idea, indeed, provides valuable insights and results for a limited range of inclusion densities, it can be easily shown that the disorder governs the response within the regime preceding the macro-failure. In fact, it has been shown (Bažant, 1989) that a self-similar evolution pattern of an initially ordered ensemble of defects represents a thermodynamically unstable path. In other words, growing disorder of micro-defects is energetically a preferred alternative. An identical conclusion was reached by Nemat-Nasser *et al.* (1982) who stated that "in reality, inclusions or voids are essentially randomly distributed and may, in fact, form clusters and other anomalous structures".

As already implied, the mean-field-based estimates of elastic moduli, derived from the first-order effective continua theories, are presumed to be valid only for the low-to-moderate inclusion volume (area) densities $f \leqslant f_{mf}$. However, in absence of requisite data for large inclusion densities $f \geqslant f_{mf}$ it is not possible to ascertain:

(a) either the upper limit f_{mf} of the range of validity of mean field models, or
(b) the relative accuracies of various competing and often contradicting mean field models available in the literature.

A very suitable framework for the analyses of critical phenomena in disordered systems is provided by one of the newly developing branches of statistical physics known as percolation theory. A system is said to percolate if it undergoes a "transition from a state of local connectedness to one in which the connections extend indefinitely" (Essam, 1980). The principal objective of percolation studies is to examine and formulate universal laws governing the behavior of the system near the transition point.

From the current viewpoint, percolation threshold will be defined as a second-order phase transition from a state defined by a diffuse, random distribution of isolated small defects (missing bonds) to a state dominated by a large (spanning) defect cluster representing loss of connectivity over a long range, i.e. spanning a large part of the specimen. The objective can, therefore, be defined as an inquiry into the behavior of a solid in the vicinity of the critical micro-defect concentration. Even more specifically, the emphasis will be placed on the functional dependence existing between the elastic (order) parameters of the material and the void concentration near the onset of the percolation threshold.

The early percolation models were primarily of lattice type. A lattice model approximates a solid by a topologically regular network of sites (nodes) interconnected by bonds. The disorder is introduced superimposing on the regular node (site) geometry a random bimodal function assigning zero strength to some of the links (bonds). Once a given initial random damage is introduced the macro-response is determined using conventional deterministic methods for analyses of articulated structures (trusses or frames). The analyses are repeated as the system is further diluted by removal of links until macro-failure is reached.

The independent variable in these analyses is the fraction (density) of ruptured (or cut) bonds p. As a lattice is gradually diluted the fraction of extant bonds increases from $p = 0$ (virgin, undamaged lattices) to $p = p_c$ (percolation threshold) and possibly to $p > p_c$ in strain-controlled experiments. At the outset $p \approx 0$ the defects (ruptured links) are small and well distributed over the entire lattice. As the number of ruptured links increases some of the defects grow and coalesce with adjacent defects forming defect clusters of size $L(p)$ having irregular shapes. Coincidentally, the elastic modulus $E(p)$ (or more accurately the macro-stiffness of the lattice) decreases. At the percolation threshold $p = p_c$ an infinite cluster of ruptured bonds ($L \to \infty$) emerges in an infinite lattice. Simultaneously, the elastic modulus (i.e. macro-stiffness) on the macro (lattice) scale vanishes.

The macro-response features two distinctly different regimes. Away from the percolation threshold the functional dependence of transport properties on p emphasizes gradual changes in concert with the basic tenets of the mean-field theories. In the close vicinity of the critical phenomena the transport properties change very slowly with increasing lacunity and are described by a power of the proximity parameter $(p_c - p)^q$.

The percolation threshold p_c is dependent only on the selected lattice, equilibrium properties near the critical point and the dimensionality d of the problem. In other words, the onset of critical phenomena depends on the details of the micro-structure of the solid. The percolation exponent q defines the behavior of the system in the vicinity of the percolation threshold and depends on the dimensionality d but not on the details of the lattice structure. This fact can be readily appreciated from the argument offered by Aharony (1986). Consider the evolution of the correlation (connectivity or coherence) length ξ defined as a root mean square distance between pairs of bonds (sites) belonging to the same defect cluster. At the percolation threshold $\xi \to \infty$ and the only scale length existing heretofore is eliminated. The system becomes qualitatively self-similar on all scales suggesting fractal nature of the phenomenon. Consequently, the details of the lattice become irrelevant at the percolation threshold giving rise to the universality of the scaling law. This fact, naturally, gives credence and significance to the simple lattice model which can, otherwise, be prematurely dismissed in comparison with more sophisticated and elaborated discretization schemes. In summary, percolation thresholds and exponents defining scaling laws are fundamental constants robust to details of microstructural texture and attendant interactions.

The heterogeneity of the material, disorder level and, thus, the range of the applicability of different analytical models directly depend on the relative magnitude of the specimen size L, size of the RVE $l \ll L$ and the correlation length ξ. A material is statistically homogeneous, rendering averaging over the RVE useful, if $l \gg \xi$ (Hansen, 1990). However, at the percolation threshold $\xi \gg l$, since $\xi \simeq L$. Consequently, within this range the RVE itself ceases to be statistically representative of the material.

It is important to emphasize that the values for the percolation threshold and percolation exponents pertain to infinite lattices, i.e. very large specimens. The corresponding

values for finite lattices can be computed using finite size scaling based on the premise that the correlation length ξ and the discretization distance separating nodes of the lattice are the only two length scales characterizing the response (Hansen *et al.*, 1989).

A higher order of stochastic geometry is considered within the framework of continuum percolation. The methods of continuum percolation are applied to investigate transport processes in topologically irregular media containing objects (inclusions, defects) of random shapes, sizes, orientations and locations. These objects may be impermeable (such as rigid inclusions) or permeable (intersecting cracks or overlapping voids).

In summary, it is apparent that the first-order (mean-field) micro-mechanical models and percolation models complement each other in terms of their application ranges. The first class of models is very useful in analyses of the response in the pre-critical (hardening) regime while the second one becomes very efficient in the proximity of the critical phenomena. Percolation models fully account for the randomness characterizing the problem which becomes the dominant feature as the defect concentration approaches critical value.

ELASTIC PLATE CONTAINING CIRCULAR CAVITIES

The objective of this study is focused on determination of the macro-stiffness for a plate made of perfectly elastic, isotropic and homogeneous material weakened by a large number of randomly distributed circular voids being small in comparison with the specimen size.

(a) *Mean-field estimates for effective elastic parameters*

The mean-field (first-order effective continuum) estimates for the elastic moduli of a perforated plate have been suggested by a host of authors in the past. In view of the extensive literature on this problem anything but a direct listing of the results and their sources seems redundant. The expressions for the elastic (Young's) modulus and the Poisson's ratio for some of the mean-field models may be readily derived from the formulas in Vavakin and Salganik (1975) and Nemat-Nasser and Hori (1990):

$$E^{DC}/E_0 = 1/(3f+1) \qquad \nu^{DC}/\nu_0 = (1+f\nu_0^{-1})/(1+3f) \tag{1}$$

$$E^{SC}/E_0 = 1-3f \qquad \nu^{SC}/\nu_0 = (1-3f)+f/\nu_0 \tag{2}$$

$$E^{DS}/E_0 = \exp(-3f) \qquad \nu^{DS}/\nu_0 = \exp(-3f) \tag{3}$$

$$E^{MDS}/E_0 = \exp(-3f) \qquad \nu^{MDS}/\nu_0 = [1-(3\nu_0)^{-1}]\exp(-3f)+(3\nu_0)^{-1} \tag{4}$$

where the superscripts DC, SC, DS and MDS stand for the dilute concentration, self-consistent model, differential scheme, and modified differential scheme, respectively. Subscript "0" indicates reference to the undamaged (virgin) plate. The formulas (1)–(4) are derived for the case of plane stress assuming that the stress on the boundary of RVE is prescribed. Additionally,

$$f = \sum (A_{void}/A) = \pi N\langle a^2\rangle \tag{5}$$

is the void surface area density, with N being the number of cavities per unit area, and A_{void} the surface area of a single void of radius a. The angular brackets denote average values.

A solution for a periodic array of circular voids allowing for their interaction (subject to a polynomial approximation for eigenstrains) but stopping short of cluster formation was derived by Nemat-Nasser *et al.* (1982).

(b) *Percolation theory estimates of effective elastic parameters*

In a general case of randomly distributed voids it is necessary to make a distinction between two different classes of problems. In the first case the voids may be considered "rigid"; i.e. two adjacent voids may only touch without intersecting or permeating each other. In the second case the cavities may overlap forming clusters of complex geometries

with serrated surfaces. Both alternatives were observed experimentally and are, therefore, deserving of attention.

Consider first the case of impermeable cavities assuming for the moment that their centers are located on a regular lattice. The critical void surface area f_c can be determined directly using either a site or a bond percolation model

$$f_c = p_c^s v = p_c^b z \tag{6}$$

where v and z are the packing (filling) factor and the coordination or connectivity number (number of closest nodes), respectively. Also p_c^s and p_c^b denote the critical fraction of sites and/or bonds occupied by voids at the percolation threshold at which the elastic modulus E vanishes. It was shown by Scher and Zallen (1970) that f_c is a dimensional invariant found to be equal to 0.45 ± 0.03. Table 1 (Appendix), distilled from a similar table in Zallen (1983), provides data necessary for the requisite computations. Observing that the product $(p_c^s v)$ (last column in Table 1) is invariant of lattice geometry, the first of the two equalities in (6), in conjunction with (5), can be rewritten (Balberg, 1987) in the form

$$f_c = (N A_{void})_c \approx 0.45. \tag{7}$$

The critical lacunity f_c in (7) is invariant to variations in N and A_{void}.

It is interesting that the elastic modulus E will, in case of nonintersecting (impermeable) voids, reduce to zero value when fewer than half of the overall surface is occupied by voids (7). To put this result into proper perspective consider a conventional cell model in which voids occupy nodes of a triangular lattice. A two-dimensional continuum is, in the spirit of the cell model, divided into identical hexagonal cells covering the entire plate. The failure according to the cell model will take place at full packing, i.e. at the site occupancy of $p_c = 1$ overestimating the critical lacunity by a factor of two. The same conclusion is valid in the case of periodic void arrays.

The situation is somewhat more complicated in the case of intersecting (permeable or soft) voids. To address this problem it is first necessary to determine the probability of overlapping of neighboring circles. Percolation threshold will then coincide with the emergence of an infinite cluster (chain) of overlapping voids. The centers of these voids need not occupy nodes of a regular lattice. Two voids of identical radius a will intersect if the distance separating their centers does not exceed value $2a$. As shown in Shante and Kirkpatrick (1971) the probability that a point, selected at random, is not within one of the circular voids is equal to $\exp(-n)$, where n is the mean number of circles within distance a from that point. At a critical concentration of circles $n = z p_c^s / 4$ (Shante and Kirkpatrick, 1971). Thus, the critical fractional surface area of voids for a two-dimensional case is (Balberg, 1987):

$$f_c = 1 - \exp(-B_c/4) \tag{8}$$

where $B_c = \lim (p_c^s z)$ (as $z \to \infty$) is the average critical number of bonds per site. The parameter B is the mean number of circle centers within a given circle, i.e. a measure of the cluster connectivity.

The critical connectivity $B = B_c$ can be estimated considering the site problem on regular lattices assuming that each site is connected (bonded) to sizes of all coordination groups. For example, in a square lattice (Table 2, Appendix) if the site is connected only to the sites of the first coordination group (nearest neighbors) $z_1 = 4$ and $z_1 p_{c1}^s = 2.36$. If the same site is connected to the sites of first two coordination groups then $z_2 = 8$ and $z_2 p_{c2}^s = 3.28$. If the same site is connected to all sites of first three coordination groups $z_3 p_{c3}^s = 3.50$ (Efros, 1986). The increment becomes smaller with addition of each new coordination group (i.e. the attractive force between sites decays with distance) and the results saturate for $z \to \infty$ at $B_c \simeq 4.55$ for all two-dimensional lattices. Thus, the critical fractional surface area of voids is, from (8), $f_c \simeq 0.68$ (Shante and Kirkpatrick, 1971; Pike and Seager, 1974).

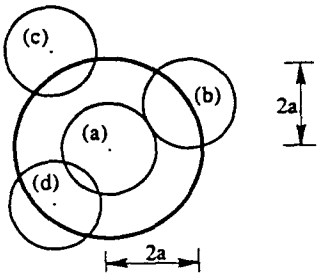

Fig. 1. Excluded area (solid line) for a circular cavity (a).

A geometrical representation of the parameter B_c was provided by Balberg *et al.* (1984) in the form

$$B_c \simeq N_c \langle A_{ex} \rangle \tag{9}$$

where $\langle A_{ex} \rangle$ is the mean excluded area of a cavity and N_c the critical number of cavities. The excluded area of an object (a) is enveloped by the locus of points formed by the centers of all surrounding geometrically similar objects (b) which just touch the object (a) without intersecting it (Fig. 1). In other words, if a center of an adjacent object (b) is within the excluded area of the object (a) two objects penetrate (permeate or intersect) each other. Thus, B_c can also be interpreted as a mean number of intersections of objects (Pike and Seager, 1974; Guyon *et al.*, 1987).

The universality of the parameter B_c and the critical fractional area f_c (8) was first demonstrated by Pike and Seager (1974) by an extensive program of numerical simulations. The value of $f_c \simeq 0.68$ was indeed found to be valid for circular voids of uniform and nonuniform radii and even for voids in the shape of squares. Once the value of f_c is known, the excluded area can be found in the case of circular voids (Fig. 1) as

$$\langle A_{ex} \rangle \simeq \pi(2a)^2 = 4A_{void} \tag{10}$$

where a is the radius of the void having surface area A_{void}.

Thus, the expressions (8), (9) and (10) with $f_c = 0.68$ lead to $N_c A_{void} = 1.14$ which, by a significant margin, exceeds the value $f_c = 0.45$ computed for the case of the non-intersecting cavities (7). Naturally, the critical surface area of circles is larger when the voids overlap. In other words, more perforations must be punched out if they are allowed to overlap. Having two different percolation thresholds is by no means contradictory. These two numbers correspond to two different processes: one characterized by separate circular cavities and other in which cavities intersect before the percolation takes place. The choice of f_c will depend on the considered material and circumstances.

The experimental data for the percolation threshold were provided by Benguigui (1986a, b) who measured the critical fraction of the removed area to be:

$f_c \approx 0.47$ for circular holes punched on a square lattice, and

$f_c \approx 0.64$ for randomly placed circular holes,

using a 0.2 mm thick copper sheet specimen subjected to simple shear. The critical value of the parameter $f = f_c$ was determined weighing the metal pieces removed by punching. These experimental measurements are in satisfactory agreement with the above-cited numerical simulations performed by Pike and Seager (1974).

The determination of the scaling law for the elastic parameters near the percolation limit ($f \to f_c$) involves considerations of the bond-bending and central forces model for the lattice, or the node-link-blob (swiss cheese) for the continuum case. These analyses (Feng

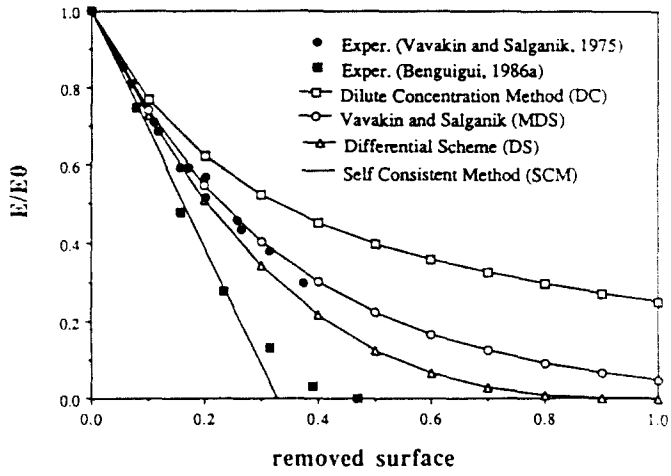

Fig. 2. Effective elastic modulus for a plate with random circular voids.

et al., 1984; Halperin *et al.*, 1985; Sornette, 1987) indicate that the tangential elastic modulus scales as

$$E \sim (f_c - f)^T. \tag{11}$$

In two dimensions the critical exponent T was found to be $T_L = 3.3 \pm 0.5$ for the lattice (Feng *et al.*, 1984) and $T_C = T_L + 3/2$ for the continuum percolation (Halperin *et al.*, 1985), indicating very slow change of the elastic modulus (macro-stiffness) near the critical point. It is perhaps reasonable to underline the fact that the elastic modulus in the above expression actually represents the stiffness of the specimen. Since the averages become meaningless in the absence of the RVE the macro-properties at this point relate to the entire specimen.

(c) *Numerical results and comparison of models*

The results of computations based on the mean-field models (expressions 1 to 5) are plotted in Fig. 2. The experimental data by Vavakin and Salganik (1975) and Benguigui (1986b) are shown in Fig. 2. Since the analytical and experimental data must be compared, it seems important to discuss the respective experimental procedures. Vavakin and Salganik (1975) used $14 \times 8 \times 0.1$ cm vacuum sheet rubber specimen, while Benguigui (1984, 1986a, b) tested $20 \times 21 \times 0.02$ metallic sheet plate. In both cases holes 1 cm in diameter were punched randomly with centers on a square grid. Benguigui (1986b) used a much more sophisticated device allowing measurements very close to the percolation limit ($f = 0.95 f_c$). Vavakin and Salganik (1975) managed as far as $f = 0.78 f_c$. It is important to note that the Benguigui (1984, 1986a) data had to be adjusted, as suggested by Sen and Thorpe (1985), to avoid confusion between the void area and void number fraction. Benguigui (1986b) reports the value of $T_L = 3.4 \pm 0.5$ for the critical exponent in (11) which matches the theoretical predictions of Feng *et al.* (1984) with a remarkable accuracy.

To facilitate the task of comparing the disparate data displayed in Fig. 2, it must be remembered that the objective of this paper is limited to the evaluation of the macro-moduli for a plate with *random* distribution of circular voids of identical size. Thus, the elastic modulus (i.e. the macro-stiffness of the specimen) *must* vanish at the percolation threshold $f = f_c$. Note that critical porosity f_c does not depend on the void shape. The presented evidence seems to favor the SCM estimate connected to the percolation threshold by a cross-over (switching or meshing) curve. The difficult problem of the determination of the cross-over curve will be addressed in a future study.

This conclusion is further supported by the related (but not identical) tests by Sieradzki and Li (1986) who used a 2 mm thick aluminum plate with predrilled holes centered on a triangular lattice. Their specimen was subsequently weakened, cutting ligaments between the holes and measuring the elastic modulus after each cut (a bond percolation problem).

The conclusion that the elastic modulus behavior away from the percolation threshold is "in good agreement with effective medium theories" is understated in the sense that the deviation of the measured data from the mean-field linear relationship is imperceptible within the range of $1.0 > E/E_0 > 0.1$. Similar results were obtained by Feng and Sen (1984) and Allain *et al.* (1985).

SUMMARY AND CONCLUSIONS

The present study advocates the application of the percolation theory in analyses of the behavior of solids in the vicinity of the critical state defined herein as the void area density at the vanishing tangential elastic modulus. The universality of the system behavior at the critical point allows for the determination of the scaling laws for the elastic moduli for large void area densities. These results nicely complement the analyses based on traditional mean-field (effective continua) models and provide, coincidentally, a measure of accuracy of these models for larger void densities.

The results and conclusions of this study are bound to stir some controversy. The reputation of the SCM has recently been on the decline. Yet, as this and other studies (Krajcinovic and Basista, 1991) demonstrate, the SCM estimates seem to be sufficiently accurate in the case of random micro-structures over a surprisingly large range of defect densities.

Despite the apparent ease with which the percolation theory can be applied to this, previously unresolved, problem of solid mechanics, a word of caution should be added in the summary. Most of the currently available data pertain to two-dimensional systems. The direct interaction between inhomogeneities is emphasized by decreasing dimensionality. Thus, the conclusions based on this study are not directly transferable to three-dimensional systems. Furthermore, the percolation thresholds with regard to conductivity and rigidity coincide in two dimensions. In three dimensions a cluster will first traverse a specimen (creating a bore-hole) before it transects it into two or more fragments. Thus, an extensive computational effort must be initiated before statistically significant data for the critical void volume density rendering $E = 0$ becomes available. It is, indeed, rather difficult to say whether critical porosity exists at all. Deptuck *et al.* (1985) report percolation thresholds at porosities of 94% in sintered silver-powder beams. They were able to produce beams with porosity of up to 93% which disintegrated after a 1% increase in porosity.

Acknowledgement—The authors gratefully acknowledge the financial support rendered by the U.S. Department of Energy, Division of Basic Energy Sciences, Division of Engineering and Geosciences and U.S. Army Research Office, Engineering Sciences Division, Structural Mechanics Branch research grants to the Arizona State University which made this work possible.

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APPENDIX

Table A1. Bond and site percolation thresholds for two-dimensional lattices

Lattice	p_c^b	p_c^s	z	v	zp_c^b	vp_c^s
Triangular	0.347	0.5	6	0.907	2.08	0.45
Square	0.5	0.593	4	0.785	2.00	0.47
Honeycomb	0.653	0.698	4	0.605	1.96	0.42

Table A2. Site percolation thresholds for first three coordinate groups

Lattice	z	p_c^s	zp_c^s
Triangular 1	6	0.5	3.00
Triangular 1, 2	12	0.295	3.54
Triangular 1, 2, 3	18	0.225	4.05
Square 1	4	0.593	2.36
Square 1, 2	8	0.410	3.28
Square 1, 2, 3	12	0.292	3.50
Honeycomb 1	3	0.698	2.10
Honeycomb 1, 2, 3	12	0.292	3.50